The Hagedorn temperature and partition thermodynamics

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Dedicated to Rolf Hagedorn, 1919–2003

Abstract. We review the resonance gas formalism of hadron thermodynamics and recall that an exponential increase of the resonance spectrum leads to a limiting temperature of hadronic matter. We then show that the number $p(n)$ of ordered partitions of an integer n grows exponentially with n and satisfies the integer counterpart of the statistical bootstrap equation. Considering the set of all partitions as a Gibbs ensemble provides a partition thermodynamics which is also governed by a limiting temperature, determined by the combinatorial structure of the problem. Further associating intrinsic quantum numbers to integers results in a phase diagram equivalent to that found in QCD for hadronic matter as a function of temperature and baryochemical potential.

1 Introduction

The most important observation in particle physics around 1960 was that the number of different species of so-called elementary particles seemed to grow without limit. Hadron–hadron collisions produced more and more resonant hadronic states of increasing masses. This phenomenon triggered two different theoretical approaches.

The more conventional idea, classical reductionism as pursued in atomism since antiquity, proposed that there must be a smaller number of more elementary objects, which then bind to form the observed hadrons as composite states. This approach, as we know, ultimately led to quantum chromodynamics as the fundamental theory of strong interactions and thus once more proved most successful.

A second, truly novel approach asked what such an increase of states would lead to in the thermodynamics of strongly interacting matter [1]. Both the question and the answer: the existence of an ultimate temperature of hadronic matter, are the contribution of Rolf Hagedorn. We know today that strong interaction thermodynamics leads to critical behavior, to a phase transition in which hadronic matter turns into a plasma of deconfined quarks and gluons. Statistical QCD has confirmed this and led to a detailed picture of this novel phenomenon. Let us see how it comes about and what is its hadronic basis.

2 Hadron thermodynamics

Consider an ideal gas of identical neutral scalar particles of mass m_0 contained in a box of volume V, assuming Boltzmann statistics. The grand canonical partition function of this system is given by

$$\mathcal{Z}(T, V) \tag{1}$$
$$= \sum_N \frac{1}{N!} \left[\frac{V}{(2\pi)^3} \int \mathrm{d}^3 p \, \exp\left\{ -\sqrt{p^2 + m_0^2} \, / T \right\} \right]^N,$$

leading to

$$\ln \mathcal{Z}(T, V) = \frac{V T m_0^2}{2\pi^2} \, K_2\left(\frac{m_0}{T} \right). \tag{2}$$

For temperatures $T \gg m_0$, the energy density of the system becomes

$$\epsilon(T) = -\frac{1}{V} \, \frac{\partial \ln \mathcal{Z}(T, V)}{\partial (1/T)} \simeq \frac{3}{\pi^2} \, T^4, \tag{3}$$

the particle density

$$n(T) = \frac{\partial \ln \mathcal{Z}(T, V)}{\partial V} \simeq \frac{1}{\pi^2} \, T^3, \tag{4}$$

and the average energy per particle

$$\omega \simeq 3 \, T. \tag{5}$$

Hence an increase of energy of the system has three consequences: it leads to
(1) a higher temperature,
(2) more constituents, and
(3) more energetic constituents.

If we now consider an interacting gas of basic hadrons and introduce resonance formation as the fundamental feature of their dynamics, we can approximate the interacting medium as an ideal gas of the possible resonance species [2]. The partition function of this resonance gas is

$$\ln \mathcal{Z}(T, V) = \sum_i \frac{VT m_i^2}{2\pi^2} \, \rho(m_i) \, K_2\left(\frac{m_i}{T}\right), \qquad (6)$$

where the sum begins with the stable ground state m_0 and then includes the possible resonances $m_i, i = 1, 2, \ldots$ with weights $\rho(m_i)$ relative to m_0. Clearly the crucial question here is how to specify $\rho(m_i)$, how many states there are of mass m_i. It is only at this point that hadron dynamics enters.

The simplest answer is that $\rho(m)$ is obtained just from combinatorics, and it immediately gives us Hagedorn's statistical bootstrap model. Hagedorn assumed that "fireballs consist of fireballs, which consist of fireballs. . . ". In a more modern form, one would assume that resonance formation and decay follow a self-similar pattern. In any case, the defining equation for $\rho(m)$ is

$$\rho(m, V_0) = \delta(m - m_0)$$

$$+ \sum_N \frac{1}{N!} \left[\frac{V_0}{(2\pi)^3}\right]^{N-1}$$

$$\times \int \prod_{i=1}^N \left[\mathrm{d}m_i \, \rho(m_i) \, \mathrm{d}^3 p_i\right] \delta^4(\Sigma_i p_i - p). \quad (7)$$

It was solved analytically by Nahm [3], giving

$$\rho(m, V_0) = \text{const.} \ m^{-3} \exp\{m/T_\mathrm{H}\}. \qquad (8)$$

The density of states thus increases exponentially in m, with a coefficient T_H^{-1} determined by

$$\frac{V_0 T_\mathrm{H}^3}{2\pi^2} (m_0/T_\mathrm{H})^2 K_2(m_0/T_\mathrm{H}) = 2\ln 2 - 1, \qquad (9)$$

in terms of the two parameters V_0 and m_0. Hagedorn assumed that the composition volume V_0, specifying the intrinsic range of the strong interactions, is determined by the inverse pion mass as scale, $V_0 \simeq (4\pi/3)m_\pi^{-3}$. This leads to a temperature $T_\mathrm{H} \simeq 150$ MeV. It should be emphasized, however, that this is just one possible way to proceed. In the limit $m_0 \to 0$, (9) gives

$$T_\mathrm{H} = [\pi^2(2\ln 2 - 1)]^{1/3} \, V_0^{-1/3} \simeq 1/r_h, \qquad (10)$$

where $V_0 = (4\pi/3)r_h^3$, and r_h denotes the range of strong interactions. With $r_h \simeq 1$ fm, we thus have $T_\mathrm{H} \simeq 200$ MeV. From this it is evident that the Hagedorn temperature persists in the chiral limit and is in fact only weakly dependent on m_π, provided the strong interaction scale V_0 is kept fixed.

If we now replace the sum in the resonance gas partition function (6) by an integral and insert the exponentially growing mass spectrum (8),

$$\ln \mathcal{Z}(T, V) \simeq \frac{VT}{2\pi^2} \int \mathrm{d}m \, m^2 \rho(m_i) \, K_2\left(\frac{m_i}{T}\right) \qquad (11)$$

$$\sim V \left[\frac{T}{2\pi}\right]^{3/2} \int \mathrm{d}m \, m^{-3/2} \exp\left\{-m\left[\frac{1}{T} - \frac{1}{T_\mathrm{H}}\right]\right\},$$

we obtain a divergence for all $T > T_\mathrm{H}$: in other words, T_H is the ultimate temperature of hadronic matter. In contrast to what we found above, an increase of energy now leads to
(1) a fixed temperature limit, $T \to T_\mathrm{H}$,
(2) the momenta of the constituents do not continue to increase, and
(3) more and more species of ever heavier particles appear.

We thus obtain a new, non-kinetic way to use energy, increasing the number of species, not the momentum per particle.

Hagedorn originally interpreted T_H as the highest possible temperature of strongly interacting matter. Somewhat later Cabibbo and Parisi [4] pointed out that the resonance gas partition function (11) in fact does not diverge at $T = T_\mathrm{H}$, while its higher derivatives do. Such a behavior occurs at phase transition points, and so T_H is "only" a critical temperature of strongly interacting matter. It is clear now that T_H indeed defines the transition from hadronic matter to a quark–gluon plasma. Hadron physics alone can only specify its inherent limit; to go beyond this limit, we need QCD.

The crucial feature leading to the observed limit of hadron physics at T_H is the exponential increase in the number of hadronic states. In the following section, we want to study the origin of such an increase and look somewhat closer at the nature of T_H.

3 Partition thermodynamics

To arrive at the simplest possible problem leading to an exponentially increasing number of "states", we consider the number $p(n)$ of ordered partitions of an integer n into integers. To illustrate: for $n = 3$, we have the ordered partitions 3, 2+1, 1+2, 1+1+1, so that

$$p(n = 3) = 4 = 2^{n-1}. \qquad (12)$$

Similarly, $n = 4$ gives the partitions 4, 3+1, 2+2, 2+1+1, 1+3, 1+2+1, 1+1+2, 1+1+1+1, leading to

$$p(n = 4) = 8 = 2^{n-1}. \qquad (13)$$

This solution can be shown to hold in fact for all n (see the appendix), i.e.,

$$p(n) = 2^{n-1} = \frac{1}{2} \, \exp\{n \ln 2\}. \qquad (14)$$

We note here for completeness that calculating the number $q(n)$ of unordered partitions of an integer n (i.e., not counting permutations) is more difficult and solved only asymptotically [5]:

$$q(n) = \frac{1}{4\sqrt{3} \, n} \mathrm{e}^{\left\{\pi\sqrt{2n/3}\right\}} \left[1 + O\left(\frac{\log n}{n^{1/4}}\right)\right]. \qquad (15)$$

We will not make use of this result here.

The problem of ordered partitions can also be solved for restricted or extended cases. For example, if we allow only odd integers in the partitions, we get for large n

$$p(n) \simeq \frac{\kappa}{\sqrt{5}} \, e^{n \ln \kappa}, \tag{16}$$

with $\kappa = (1 + \sqrt{5})/2 \simeq 1.62 < 2$. On the other hand, if we introduce further degrees of freedom, e.g., giving each integer an intrinsic quantum number ("spin") which can take on the values ± 1, the exact solution is

$$p(n) = 2\kappa^{n-1} = \frac{2}{\kappa} \, e^{\{n \ln \kappa\}}, \tag{17}$$

now with $\kappa = 3 > 2$. Hence we have as solution to the generalized ordered partition problem the form (see the appendix)

$$p(n) \propto \kappa^n = e^{\{n \ln \kappa\}}, \tag{18}$$

with $\kappa = 2$ for the "standard" case and in general $\infty > \kappa > 1$.

We thus obtain an exponential increase in the number of possible partitions. How is this related to the statistical bootstrap condition (7)? Consider a density of states defined by a bootstrap equation for integers,

$$\rho(n) = \delta(n-1) + \sum_{k=2}^{n} \frac{1}{k!} \prod_{i=1}^{k} \rho(n_i) \, \delta(\Sigma_i n_i - n). \tag{19}$$

Its solution is just the number of partitions of n,

$$\rho(n) = z \, p(n) \tag{20}$$

up to a normalization of order unity (for the standard case $\kappa = 2$, $z \simeq 1.25$). We can thus conclude
(1) the statistical bootstrap equation is an integral formulation of a partition problem;
(2) the exponential increase in the number of states is of combinatoric origin, and
(3) the exponential coefficient κ is determined by the combinatoric structure.

Let us therefore consider κ in more detail. In the standard case, see (14), the exponential increase in the number of partitions is determined by $\ln 2$; in (9), we have seen that in the actual statistical bootstrap, T_H is also determined by $\ln 2$, but in a somewhat more complex way due to the presence of momentum degrees of freedom in addition to the hadron masses.

To understand the meaning of κ in a thermal context, we construct a statistical mechanics of partitions, considering each partition as a point in the "phase space" of all partitions of fixed n. The set of all partitions thus forms a Gibbs ensemble, and we define the partition entropy as

$$S(n) = \ln p(n) \simeq n \ln \kappa, \tag{21}$$

the specific partition entropy as

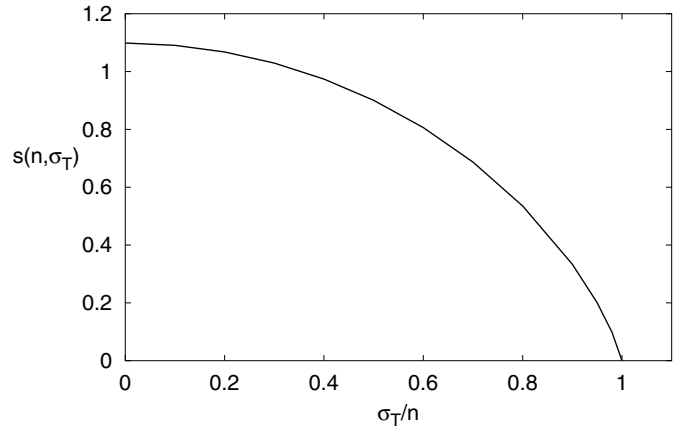$$s(n) = \frac{S(n)}{n} \simeq \ln \kappa, \tag{22}$$

Fig. 1. Specific partition entropy as function of $x = \sigma_T/n$

and the partition temperature Θ as

$$\frac{1}{\Theta} = \frac{dS(n)}{dn} = \ln \kappa. \tag{23}$$

The parameter $\ln \kappa$ determining the increase of the number of partitions of n is thus the partition temperature Θ, which here for large n coincides with the inverse of the specific partition entropy.

Next we turn to the mentioned generalized form, in which each integer n_i has an intrinsic "spin" variable $\sigma_i = \pm 1$, in order to see what the effect of conserved quantum numbers is. We define the total spin for a given partition of n as

$$\sigma_T(n) = \sum_i \sigma_i, \tag{24}$$

where the sum runs over all $k \le n$ terms of the partitions. We now calculate the number $p(n, \sigma_T)$ of partitions of n at fixed σ_T. For $\sigma_T = n$ we have $p(n, n) = 1$, since only the partition $n = 1 + 1 + \ldots + 1$ of all spins $+1$ can give a total spin equal to n. For $\sigma_T = 0$ we find $p(n, 0) \simeq 3^{n-1}$, since total spin zero constitutes the dominant part of all configurations. The specific partition entropy $s(n, \sigma_T)$ is shown in Fig. 1. It was calculated numerically for values of n up to 20000 (see the appendix) and it was found to depend only on $x = \sigma_T/n$. Moreover, our calculations show no appreciable difference between $n = 10000$ and $n = 20000$, so that the latter is in good approximation the asymptotic form of the curve. It can be fitted by

$$s(x) = [1 - x^a]^b \ln 3, \tag{25}$$

with $a = 2.1, b = 0.736$, and is shown in Fig. 1, varying from $s(0) = \ln 3$ to $s(1) = 0$.

The partition temperature now becomes

$$\frac{1}{\Theta(x)} = \frac{dS(n, \sigma_T)}{dn} = \frac{d[n \, s(x)]}{dn} = s(x) - x \frac{ds(x)}{dx} \tag{26}$$

and is thus no longer equivalent to the inverse specific partition entropy: for $x \to 1$, both $s(x)$ and $\Theta(x)$ vanish. Through (26) we can determine $\Theta(x)$ from the function $s(x)$ (see (25)) that we have previously calculated. The resulting
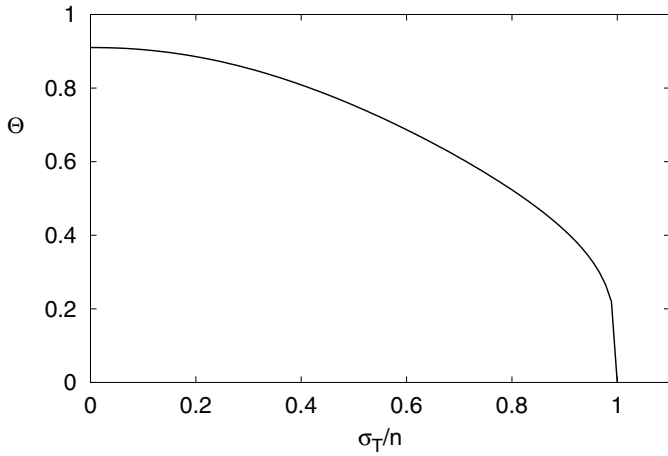
Fig. 2. Partition temperature as a function of $x = \sigma_T/n$

form is shown in Fig. 2; it decreases from $\Theta = 1/\ln 3$ at $x = 0$ to 0 at $x = 1$.

The constituents of our partition thermodynamics are the different integers occurring in the partitions; they are the resonances of the resonance gas. It is thus interesting to ask for the relative abundance $R_{n,\sigma_T}(a,b)$ of two integers $a, b < n$ in the set of all partitions of n. In the appendix, we calculate the total occurrence $h(n, a, \sigma_T)$ of an integer a in the partitions of n at fixed σ_T. Given this, the desired relative abundance is defined as

$$R_{n,\sigma_T}(a,b) = \frac{h(n, a, \sigma_T)}{h(n, b, \sigma_T)}. \tag{27}$$

For $a, b \ll n$ it turns out that for large n, $R_{n,\sigma_T}(a,b)$ depends only on the ratio σ_T/n and has the form

$$R_{n,\sigma_T}(a,b) \simeq e^{-(a-b)/\Theta(\sigma_T/n)}, \tag{28}$$

where Θ is precisely the partition temperature shown in Fig. 2. We thus see that the resonance gas feature of species abundances specified by one universal temperature in fact has its origin in the underlying combinatoric structure.

So far, we have considered partition thermodynamics in what might be called a grand microcanonical formulation: the number of terms in the different partitions varies, but their sum n and the overall spin σ_T are specified exactly. To go to a grand canonical formulation, we fix n and σ_T only on the average. The starting point is the grand canonical partition function

$$\mathcal{Z}(T,\mu) \tag{29}$$
$$= \int_1^\infty dn \exp\{-n/T\} \int_{-n}^n d\sigma_T \exp\{-\mu\sigma_T/T\} \, p(n, \sigma_T),$$

with the temperature T and the chemical potential μ the Lagrangian multipliers for n and σ_T. Since $p(n, \sigma_T) = \exp\{ns(\sigma_T/n)\}$, where $s(\sigma_T/n)$ is the specific entropy calculated above, we get

$$\mathcal{Z}(T,\mu) \tag{30}$$
$$= \int_1^\infty dn \int_{-n}^n d\sigma_T \exp\left\{-n\left[\frac{1}{T} - \frac{\mu}{T}\frac{\sigma_T}{n} - s(\sigma_T/n)\right]\right\}.$$
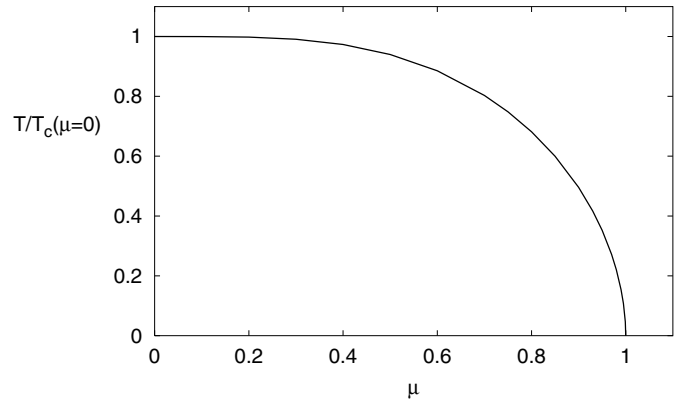

Fig. 3. Phase boundary for partition thermodynamics

The integral (30) is not defined for all pairs of values (T, μ): for fixed μ, it exists only for those values of the temperature for which

$$\frac{1}{T} - \frac{\mu}{T}\frac{\sigma_T}{n} - s(\sigma_T/n) > 0 \quad \Rightarrow \quad T < \frac{1 - \mu x}{s(x)} \quad \forall \, \sigma_T, n. \tag{31}$$

We note that the function on the right of the last inequality depends only on the variable $x = \sigma_T/n$, and not on σ_T and n separately. Since the condition (31) must be valid for any possible value of the ratio σ_T/n, with $|\sigma_T/n| \le 1$, T has to be smaller than the absolute minimum of the function

$$f(x, \mu) = \frac{1 - \mu\, x}{s(x)}, \tag{32}$$

in the interval $-1 \le x \le 1$. Hence for any μ there exists a limiting value $T_c(\mu)$ of the temperature beyond which the grand canonical partition function is not defined. In particular, for $\mu = 0$, we get $T_c = 1/\ln 3$, which is the temperature determined in the previous section for the partition problem with total spin $\sigma_T = 0$. We further note that $|\mu| \le 1$, because otherwise the function $f(x, \mu)$ would become negative for some x and the inequality (31) could never be satisfied by positive values of the temperature T. By varying μ we obtain a curve in the T–μ plane, which defines the existence region for our system. Determining numerically the minimum of the $f(x, \mu)$ as a function of μ, we obtain the form shown in Fig. 3.

The resulting "phase diagram" of partition thermodynamics, i.e., the boundary for the existence of such thermodynamic systems, is evidently very similar to that expected in statistical QCD for the phase boundary of hadronic matter. It thus seems that quantum chromodynamics leads to a resonance pattern which is basically of a combinatoric nature. Moreover, the conservation of baryon number also appears to follow a simple combinatoric form resulting from the addition of the baryonic degrees in the resonance composition.

4 Conclusions

We have shown that the limit of hadron physics obtained in resonance gas thermodynamics with an exponentially

increasing mass spectrum is due to the combinatorics of resonance formation. In particular, the simplest partition systems provide

(1) a limiting temperature determined solely by the combinatoric structure of the system;

(2) a reduction of this limiting temperature for conserved intrinsic quantum numbers and hence a phase boundary for partition thermodynamics;

(3) a relative abundance of constituents governed by the limiting temperature.

Hence, if all combinations of n hadrons are resonances and the degeneracy of each resonance is determined by the number of composition patterns, an energy input eventually goes into species formation, not into faster constituents. Comparing such a resonance gas to a conventional pion gas, we thus find for the dependence on energy density ϵ for a pion gas $n_\pi \sim \epsilon^{3/4}$ and $\omega_\pi \sim \epsilon^{1/4}$, and for a resonance gas $n_{\rm res} \sim \epsilon$ and $\omega_{\rm res} \sim$ const.

Here n denotes the density of constituents, ω the average energy of a constituent.

We close with a rather speculative question, concerning the relation of our considerations to nuclear collisions. In the experimental study of such interactions, it is found that the relative abundances of hadron species are essentially determined by two parameters, the temperature T of a resonance gas and the baryochemical potential μ specifying its overall baryon density [6]. For high energy collisions at $\mu = 0$, $T \simeq 175\,{\rm MeV}$; with increasing μ, $T(\mu)$ decreases in a form quite similar to that shown in Fig. 3. What can we learn from this? There seem to be two possible scenarios.

(1) Each collision produces a thermal system and thus corresponds to an ensemble of many partitions: nuclear collisions produce matter.

(2) Each collision is one partition, and only the ensemble over many collisions forms a thermal ensemble: nuclear collisions simulate matter.

Which of these two is realized in nature appears to be one of the most crucial questions for the experimental high energy heavy ion program.

A Standard case

To determine the number of ordered partitions of an integer n into integers $k_i \leq n$, we start from the number of partitions containing m terms. This is equivalent to the number of divisions of a segment of length n units into m parts. Each partition of this type is constructed by choosing $m-1$ of the first $n-1$ integers as end points for the m segments dividing $[0, n]$. This can be done in

$$p(n, m) = \binom{n-1}{m-1} \qquad (33)$$

possible ways. By summing over all possible values of m, we obtain

$$p(n) = \sum_{m=1}^{n} p(n, m) = \sum_{m=1}^{n} \binom{n-1}{m-1} = 2^{n-1}. \qquad (34)$$

Next, we determine the abundance of a given integer a in the set of all partitions. Suppose we have a partition of m terms with at least one occurrence of the number $a < n$. By changing the position of a within the partition, we get m possible partitions containing the same number of terms, with order of the $m-1$ other terms unchanged. Hence the number of occurrences of a in partitions with m summands is just m times the number of partitions of $n-a$ into $m-1$ terms. Among the partitions of $n-a$ into $m-1$ terms there will in general be other a's, and in this case each partition gives a contribution larger than one to the occurrence of a. However, this larger contribution is exactly compensated by a reduction of the number of corresponding partitions, since the different a are indistinguishable and can be freely permuted without generating new partitions. To get the total occurrence $h(n, a)$ of a in all partitions of n we sum over all possible values of m and obtain

$$h(n, a) = \sum_{m=2}^{n-a+1} m \binom{n-a-1}{m-2}. \qquad (35)$$

Defining $l = m - 2$, we get

$$h(n, a) = \sum_{l=0}^{n-a-1} (l+2) \binom{n-a-1}{l}$$

$$= 2 \sum_{l=0}^{n-a-1} \binom{n-a-1}{l} + \sum_{l=0}^{n-a-1} l \binom{n-a-1}{l}, \qquad (36)$$

and from this

$$h(n, a) = (n - a - 1)\, 2^{n-a-2} + 2^{n-a} = \frac{n-a+3}{4}\, 2^{n-a} \qquad (37)$$

for the abundance of a in the partitions of n.

B Intrinsic degrees of freedom

We now assume that each integer carries an intrinsic degree of freedom, an integer "spin" σ, giving it 2σ degenerate states; in the example above, we had $\sigma = 1$ and hence the two states ± 1. It is obvious that such a spin leads to an increase in the number of partitions; each partition is now labeled by n and by the total spin $\sigma_{\rm T} = \sum_k \sigma_k$ of its summands. If we do not impose any constraint on the total spin, it is trivial to calculate the total number of partitions. There are now 2σ copies of each number, so that in a partition with m terms, each summand contributes an additional factor 2σ. Equations (33) thus becomes

$$p_\sigma(n, m) = (2\sigma)^m \binom{n-1}{m-1}, \qquad (38)$$

so that

$$p_\sigma(n) = \sum_{m=1}^{n} p_\sigma(n, m) = \sum_{m=1}^{n} (2\sigma)^m \binom{n-1}{m-1}$$

$$= (2\sigma)\,(2\sigma + 1)^{n-1}. \tag{39}$$

The problem is more interesting if σ_T has a well defined value, which is equivalent to requiring the conservation of a spin, isospin or baryon number for a system of particles. In this case, it is not possible to compute the total number of partitions and the abundances in closed form. However, at least for the special case $\sigma = 1$, we can derive formulas that allow a numerical computation of the quantities of interest. Since the problem is completely symmetric under a change of sign of the spin, we consider positive values of the total spin.

We start by asking how many partitions of m terms there are with a total spin σ_T. The summands of the partitions carry spin $+1$ or -1. If there are k numbers with spin $+1$ there must then be $m - k$ numbers with spin -1. In order to have a total spin σ_T, there is only one possibility: $(m + \sigma_T)/2$ terms must carry spin $+1$ and $(m - \sigma_T)/2$ terms spin -1. Hence the required number of partitions at fixed m is

$$p(n, m, \sigma_T) = \binom{m}{(m + \sigma_T)/2}\binom{n - 1}{m - 1}, \tag{40}$$

and the total number becomes

$$p(n, \sigma_T) = \sum_{m=\sigma_T}^{n} p(n, m, \sigma_T)$$

$$= \sum_{m=\sigma_T}^{n} \binom{m}{(m + \sigma_T)/2}\binom{n - 1}{m - 1}. \tag{41}$$

Note that the sum over m is constrained by the parity of the total spin: if σ_T is even (odd), m must be even (odd). Hence the sums run over those values of m between σ_T and n which have the same parity as σ_T.

To obtain the abundances, we only have to insert in (35) the factor due to the spin degeneracy of the partitions, given in (41), leading to

$$h(n, a, \sigma_T) = \sum_{m=\sigma_T}^{n} m \binom{m}{(m + \sigma_T)/2}\binom{n - a - 1}{m - 2}. \tag{42}$$

Equations (41) and (42) can easily be evaluated using *Mathematica*.

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